PHYC 521 Fall 09

Lec 1: 08/24/2009

Review of Mathematical Foundation:

Linear Vector Spaces:

A linear vector space $\mathbb{V}$ is a set $\{V_1, V_2, V_3, \dots\}$ of vectors that yield only elements of $\mathbb{V}$ under the addition and multiplication by scalars:

$V_i + V_j \in \mathbb{V}$ $\qquad$ $\underbrace{\alpha}_{\text{scalar}} V_i \in \mathbb{V}$

Axioms for addition:

$V_i + V_j = V_j + V_i$ commutativity

$V_i + (V_j + V_k) = (V_i + V_j) + V_k$ associativity

$V_i + 0 = V_i$ 0 unique null vector

$V_i + (-V_i) = 0$ existence of inverse

Axioms for scalar multiplication:

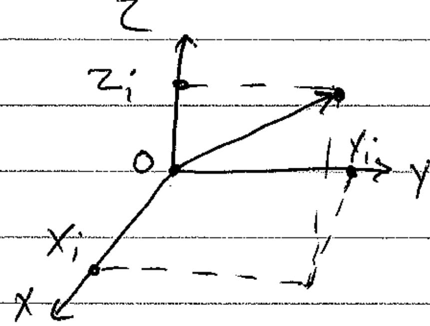
$\alpha(V_i + V_j) = \alpha V_i + \alpha V_j$

$(\alpha+\beta)\,\mathbb{V}_i = \alpha\,\mathbb{V}_i + \beta\,\mathbb{V}_i$

$\alpha(\beta\,\mathbb{V}_i) = (\alpha\beta)\,\mathbb{V}_i$

Example: All directed line segments in three dimensions: $(\mathbb{V}^3(R))$

$$\mathbb{V}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}$$



$$\mathbb{V}_i + \mathbb{V}_j = \begin{bmatrix} x_i + x_j \\ y_i + y_j \\ z_i + z_j \end{bmatrix} \qquad \alpha\,\mathbb{V}_i = \begin{bmatrix} \alpha x_i \\ \alpha y_i \\ \alpha z_i \end{bmatrix}$$

α → real number

A set of vectors $\{\mathbb{V}_1, \ldots, \mathbb{V}_n\}$ is said to be linearly independent if:

$$\sum_{i=1}^{n} \alpha_i \mathbb{V}_i = 0 \Rightarrow \alpha_i = 0 \quad \forall i$$

A vector space $\mathbb{V}$ is n dimensional if it admits at most n linearly independent vectors.

($\mathbb{V}^3(R)$ is 3 dimensional)

Inner product of two vectors is a scalar and satisfies the following axioms:

$$(V_i, V_i) \geq 0 \qquad (0 \text{ if } V_i = 0)$$

$$(V_i, V_j) = (V_j, V_i)^*$$

$$(V_i, \alpha V_j + \beta V_k) = \alpha (V_i, V_j) + \beta (V_i, V_k)$$

In $\mathbb{V}^3(C)$ we have:

$$(V_i, V_j) = [x_i^* \quad y_i^* \quad z_i^*] \begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = x_i^* x_j + y_i^* y_j + z_i^* z_j$$

complex numbers

From now on, we use Dirac notation.

$$|V_i\rangle = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \qquad \langle V_i| = [x_i^* \quad y_j^* \quad z_i^*]$$

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In this notation the inner product is $\langle V_i | V_j \rangle$.

A set of vector $\{|e_1\rangle, \dots |e_n\rangle\}$ is called orthonormal if:

$\langle e_i | e_j \rangle = \delta_{ij}$ $\quad (\delta_{ij} = 0 \text{ if } i \neq j, \delta_{ij} = 1 \text{ if } i = j)$

Kronecker delta

In $\mathbb{V}^3(\mathbb{R})$ $\vec{i}, \vec{j}, \vec{k}$ form an orthonormal set.

$$\vec{i} \equiv \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{j} \equiv \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{k} \equiv \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Length of a vector V is defined as:

$$|V| \equiv \sqrt{\langle V | V \rangle}$$

Two useful inequalities.

1- Schwarz inequality:

$$|\langle V_i | V_j \rangle|^2 \leqslant |V_i|^2 \, |V_j|^2$$

2- Triangle inequality:

$$|V_i + V_j| \leqslant |V_i| + |V_j|$$